

Effective Compton Cross Section in Non-Degenerate High Temperature Media

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February 2, 2008

Abstract

The effective compton cross section in a non-degenerate plasma ($n \ll \left\{ \frac{(kT/c)^2 + 2mkT}{h^2} \right\}^{3/2}$) is investigated in a wide range of temperatures. The results show a decreasing behavior with temperature especially for $kT \gg m_e c^2$. The results may be important in phenomena like accretion discs or ultra-relativistic blast waves in GRB models, where the emitted radiation has to pass through a medium containing high energy electrons.

keywords: Compton scattering, high energy physics, gamma-ray bursts, accretion

1 Introduction

Compton scattering is one of the most important phenomena in astrophysics that affects the received flux from terrestrial objects. In a non-degenerate low temperature plasma in which $\frac{n^{2/3} h^2}{2\pi m_e} \ll kT \ll m_e c^2$ [1], the free electrons can be considered effectively at rest, provided that the photon energy $E = h\nu$ to be much greater than kT . In such a situation the Klein-Nishina formula for Compton scattering accurately works. The Klein-Nishina formula is only dependent on the energy of the incoming photon as measured in the rest frame of the electron [2], [3] :

$$\sigma_c = \sigma_T \cdot \frac{3}{4} \left\{ \frac{1+\epsilon}{\epsilon^2} \left[\frac{2(1+\epsilon)}{1+2\epsilon} - \frac{1}{\epsilon} \ln(1+2\epsilon) \right] + \frac{1}{2\epsilon} \ln(1+2\epsilon) - \frac{1+3\epsilon}{(1+2\epsilon)^2} \right\} \quad (1)$$

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in which ϵ is the ratio of the photon energy E to the electron rest energy $m_e c^2$ (Fig.(1)). But at high temperatures where the electron mean energy is greater than or comparable to $m_e c^2$, the energy of the photon is highly different relative to different individual electrons. So, since the total Compton cross section is a function of the photon energy in the rest frame of the electrons, one would expect an effective cross section due to the thermal distribution of electrons. The aim of this paper is to obtain an effective cross section for Compton scattering in a wide range of plasma temperatures and photon energies. In Sec. 2 the averaging method is introduced. In Sec. 3 the momentum distribution of electrons in a wide range of temperatures is discussed and the normalization constant of the distribution is evaluated as a function of plasma temperature, and then, a numerically computable form for the effective cross section of Compton scattering is obtained. The results are presented in Sec. 4, as well as a discussion on their importance in GRBs.

2 Averaging formalism

Consider a monochromatic photon beam with energy $E = h\nu$ passing through a plasma that its temperature T and its electron number density n_e satisfy the non-degeneracy condition $n \ll \left\{ \frac{(kT/c)^2 + 2mkT}{h^2} \right\}^{3/2}$. Now, we define the effective cross section in Compton scattering $\sigma_{C,eff}$ as:

$$\frac{dN}{N} = \sigma_{C,eff} \cdot n_e \cdot dl \quad (2)$$

where $\frac{dN}{N}$ denotes the fraction of photons scattered in a distance dl . Now, let's consider the fraction of electrons $\frac{\delta n_e}{n_e}$ that move in the solid angle $d\Omega$ and have a momentum magnitude between p and $p+dp$. The fraction has the value:

$$\frac{\delta n_e}{n_e} = \frac{d^3 f}{d\vec{p}^3} p^2 dp d\Omega \quad (3)$$

where $\frac{d^3 f}{d\vec{p}^3}$ is the probability density for an electron to have momentum \vec{p} . Let's give attention to the photons that scatter by these electrons. Denoting them by δN , we can write :

$$\frac{dN}{N} = \frac{\int \delta N}{N} \quad (4)$$

which means that the total scattering is the result of integration over all possible scatterings by the electrons having different momenta. To find $\frac{\delta N}{N}$, we make a relativistic transformation to the comoving frame of the electrons denoted by δn_e (Eqn.(3)), hereafter called as "the prime frame". The scattering of the photons by these electrons is an "event", as it mean in the theory of relativity.

So, the quantity $\frac{\delta N}{N}$ is an invariant and once evaluated in a specific frame, its magnitude can be used in all other frames. Since the electrons denoted by δn_e are at rest in prime frame (which is their own comoving frame) we can use Klein-Nishina cross section (Eqn. (1)) to evaluate $\frac{\delta N}{N}$ in this frame:

$$\frac{\delta N}{N} = \sigma_c(\epsilon') \cdot \delta n'_e \cdot dl' \quad (5)$$

where, ϵ' , $\delta n'_e$, and dl' correspond the quantities ϵ , δn_e and dl in the the prime frame, so that :

$$\epsilon' = \epsilon \gamma (1 - \beta \cos \theta) \quad (6)$$

$$\delta n'_e = \frac{\delta n_e}{\gamma} \quad (7)$$

$$dl' = dl \gamma (1 - \beta \cos \theta) \quad (8)$$

where γ and β are the Lorentz factor and the speed of the mentioned electrons respectively and θ , the polar angle that the electrons make with the photon beam direction, all measured in laboratory frame. Using Eqs.(6) to (8), Eqn.(5) can be written as:

$$\frac{\delta N}{N} = \sigma_c(\epsilon \gamma (1 - \beta \cos \theta)) \cdot \delta n_e \cdot (1 - \beta \cos \theta) dl \quad (9)$$

and considering Eqn.(4), we see:

$$\frac{dN}{N} = \left[\int \int \sigma_c(\epsilon \gamma (1 - \beta \cos \theta)) \cdot (1 - \beta \cos \theta) \cdot \frac{d^3 f}{d\vec{p}^3} p^2 dp d\Omega \right] \cdot n_e \cdot dl \quad (10)$$

where we used Eqs.(3) and (9). Comparing Eqn.(10) with Eqn.(2) we obtain finally:

$$\sigma_{C,eff} = \int \int \sigma_c(\epsilon \gamma (1 - \beta \cos \theta)) \cdot (1 - \beta \cos \theta) \cdot \frac{d^3 f}{d\vec{p}^3} p^2 dp d\Omega \quad (11)$$

To use this expression in evaluating $\sigma_{C,eff}$, firstly we need to find $\frac{d^3 f}{d\vec{p}^3}$ explicitly. The next section is devoted to this task.

3 Normalization Constants

The probability density for an electron to have a momentum \vec{p} is proportional to the Boltzman factor $e^{-E/kT}$, so that:

$$\frac{d^3 f}{d\vec{p}^3} = C(T) e^{-E(p)/kT} \quad (12)$$

We want to find the normalization constant $C(T)$ which has the following form:

$$C(t) = \left\{ \int_0^\infty e^{-E(p)/kT} 4\pi p^2 dp \right\}^{-1} \quad (13)$$

Assuming the electrons to be free we simply have:

$$E(p) = \sqrt{p^2 c^2 + m_e^2 c^4} \quad (14)$$

It is well known that the integration has simple analytic solutions at two extreme temperature limits namely, non-relativistic ($kT \ll m_e c^2$) and ultra-relativistic ($kT \gg m_e c^2$) temperatures:

$$C(T) = \begin{cases} (2\pi m_e kT)^{-3/2} & \text{for } kT \ll m_e c^2 \\ \frac{1}{\pi} \left(\frac{kT}{c}\right)^{-3} & \text{for } kT \gg m_e c^2 \end{cases} \quad (15)$$

But at intermediate temperatures there is no analytic solution for the integration. So it must be solved numerically. The obtained numerical results are shown in Fig.(3). It can be seen that $C(t)$ has a wide range of orders of magnitude. Really in practice we had to take exact care in parameterizing the quantities, otherwise the numerical computation would have been impossible. The procedure used in parameterizing the quantities maybe worth mentioning. the range of the integration in Eqn.(13) is from zero to infinity. Therefore, we must concern ourselves with the momentum p_{max} in which the integrand $p^2 e^{-E(p)/kT}$ reaches its maximum value. This is described by the equation:

$$\frac{d}{dp} (p^2 e^{-E(p)/kT}) \big|_{p=p_m} = 0 \quad (16)$$

which has the solution :

$$p_{max}(T) = \sqrt{2} \left\{ 1 + \sqrt{1 + \left(\frac{m_e c^2}{kT}\right)^2} \right\} \frac{kT}{c} \quad (17)$$

Now, we define a number of non-dimensional parameters as the following :

$$\xi \equiv \frac{p}{p_{max}} \quad (18)$$

$$\tau \equiv \frac{kT}{m_e c^2} \quad (19)$$

$$c(\tau) \equiv C(T) p_{max}^3(T) \quad (20)$$

$$g(\tau) \equiv \sqrt{2} \tau \left\{ 1 + \sqrt{1 + \frac{1}{\tau^2}} \right\} \quad (21)$$

$$\alpha(\xi, \tau) \equiv \frac{E(p)}{kT} = \frac{1}{\tau} \{ \sqrt{1 + \xi^2 g(\tau)^2} - 1 \} \quad (22)$$

and use them to convert Eqs.(12) and (13) to non-dimensional forms below :

$$\frac{d^3 f}{d\vec{\xi}^3} = c(\tau) e^{-\alpha(\xi, \tau)} \quad (23)$$

$$c(\tau) = \left\{ \int_0^\infty 4\pi \xi^2 e^{-\alpha(\xi, \tau)} d\xi \right\}^{-1} \quad (24)$$

Considering Eqn.(12), and Eqs.(17) to (23), it can be seen that the quantity:

$$\frac{df}{d\xi} \equiv 4\pi \xi^2 \frac{d^3 f}{d\vec{\xi}^3} = 4\pi c(\tau) \xi^2 e^{-\alpha(\xi, \tau)} \quad (25)$$

is naturally normalized:

$$\int \frac{df}{d\xi} d\xi = 1 \quad (26)$$

and takes its maximum value at $\xi = 1$ which is independent of τ . Really, the parameterizing procedure was designed so that $\frac{df}{d\xi}$ reach to its maximum at a point independent of τ , otherwise the wide range of τ values would make the numerical computations impossible or at least inconvenient.

The results are shown in Fig.(2), and as expected, the non-dimensional normalization constant $c(\tau)$ is a quantity of order of one. Having these results in hands and using Eqs.(17), (19) and (20), the actual normalization constant $C(T)$ can be obtained. the results are shown in Fig.(3).

Now, all is prepared to return to Eqn.(11) and rewrite it after some algebra in terms of the defined parameters:

$$\frac{\sigma_{c,eff}}{\sigma_T} = 2\pi c(\tau) \int_{\theta=0}^{\pi} \int_0^\infty \frac{1}{\sigma_T} \sigma_c(\epsilon\gamma(1 - \beta \cos \theta)) \cdot (1 - \beta \cos \theta) \xi^2 e^{-\alpha(\xi, \tau)} \sin \theta d\theta d\xi \quad (27)$$

Noting that γ and β in this relation must be considered as functions of ξ and τ . So, let's write them firstly in terms of momentum p :

$$\gamma = \sqrt{\left(\frac{p}{m_e c}\right)^2 + 1} \quad (28)$$

$$\beta = \frac{p c}{\sqrt{p^2 c^2 + m_e^2 c^4}} \quad (29)$$

and use Eqs.(17) to (19) and Eqn.(21) to obtain:

$$\gamma = \sqrt{1 + \xi^2 g(\tau)^2} \quad (30)$$

$$\beta = \frac{\xi g(\tau)}{\sqrt{1 + \xi^2 g(\tau)^2}} \quad (31)$$

Now the numerical evaluation of Eqn.(27) is straight forward. The obtained results are presented and discussed in the next section.

4 Results and Discussion

As it can be seen in the three-dimensional plot of Fig.(4), and in the contour plot of Fig.(5), the effective cross section is closely the same as the σ_c (Fig.(1)) at temperatures $kT < m_e c^2$. This is an anticipated result. The deviation begins when $kT > m_e c^2$, with an overall shift to left at higher and higher temperatures. This behavior could have been expected too. The Compton cross section decreases by increasing of photon energy (Fig.(1)), so, for a photon with energy $E = h\nu$, the more the temperature is, the more energy in rest frame of individual electrons it has, and furthermore, the less the effective cross section is expected. This simple reasoning may seem wrong of course. Because though it is correct for the electrons moving toward the photon beam but in the rest frame of the electrons moving along the beam the energy of a photon would be less than the corresponding value as measured in laboratory frame and it decreases more and more in higher and higher temperatures, and so, in contrary one may expect the effective cross section to be less and less, as long as these electrons are concerned. Really, the final answer can be found in Eqs.(5) and (8). For ultra-relativistic electrons the distance dl' as seen by the electrons moving in the same direction as the beam one, is $4\gamma^2$ times less than the corresponding value for ones moving in the opposite direction. So, as can be seen in Eqn. (5), despite the greater cross section they have individually, their share in scattering happens to be less than that other electrons, so that in high temperatures the effective cross section as a whole is determined by the electrons moving toward the photon beam. In Fig.(6) the portion of these two groups of electrons in Compton scattering is compared with each other.

In a GRB model prepared by authors [4], a collimated ultra-relativistic ejecta with a Lorentz factor $\Gamma \sim 1000$ collides with a dense cloud surrounding the stellar engine. The photons emitted from the shocked medium has to pass through the cloud before entering free space, but the opacity of the cloud is extremely too high to let them escape. Really, in the model the cloud thickness L , and its density n , are of order $10^{13} cm$ and $10^{17} cm^{-3}$ respectively, which result in an opacity high to $\sigma_T n L \sim 10^6$ which obviously prevents any radiation to escape. But, since in external shock models for GRBs (see [5] and [6] for a review) the particles in the shocked medium are expected to have mean energies of order $\frac{u}{4\Gamma n} = \Gamma m_p c^2 \sim 10^6 m_e c^2$, the real optical depth of the medium for *Mev* photons would be 10^{-6} times less than what might be roughly expected in

beginning (Table (1)), and consequently a photon radiated from the shocked matter may succeed to go out of it without being scattered, provided that its passage make an angle larger than $(\sqrt{\frac{5}{3}}\Gamma)^{-1}$ with the velocity vector of shocked matter so that it remain in the shocked medium by the time it cross the opaque cloud. So, we concluded that if the shock front succeed to cross the cloud it might be seen and so might really make a GRB, and if it do not and stop in the cloud, its photons would not be able to cross the cool dense cloud and the phenomenon must be considered a *failed* GRB. Though the presented reasoning may seem to be restricted only to a medium in thermal equilibrium but it must be correct for all media in which the mean energy of electrons is of order of $10^6 m_e c^2$. for example in GRB models, the electrons in the shocked matter are assumed to have a power law distribution:

$$N(\gamma_e) \propto \gamma_e^2 \quad \text{for :} \quad \gamma_e > \gamma_{e,min} \quad (32)$$

By repeating the averaging procedure introduced in this paper the effective Compton cross section for such a distribution can be found [7]. The obtained results are comparable in orders of magnitude to ones presented in Table (1) and shown in Figs.(4) and (5), replacing τ with mean Lorentz factor of electrons.

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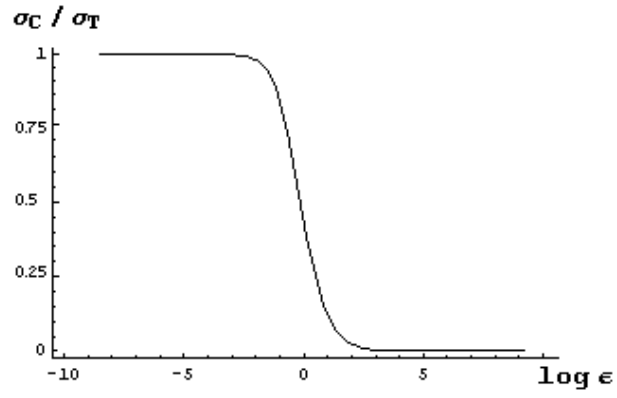


Figure 1: the cross section in Compton scattering evaluated using Eqn.(1)

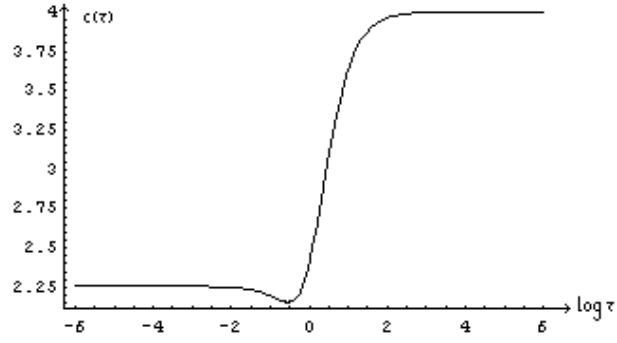


Figure 2: The parameterized non-dimensional normalization constant $c(\tau)$ verses $\log \tau$

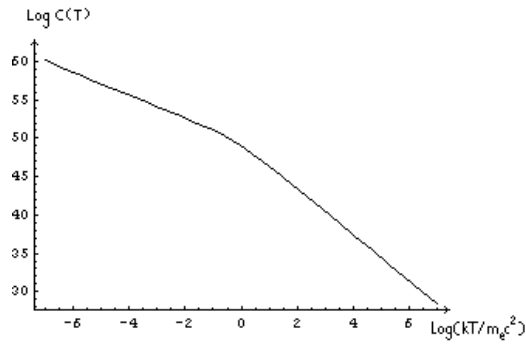


Figure 3: The normalization constant $C(T)$ verses temperature

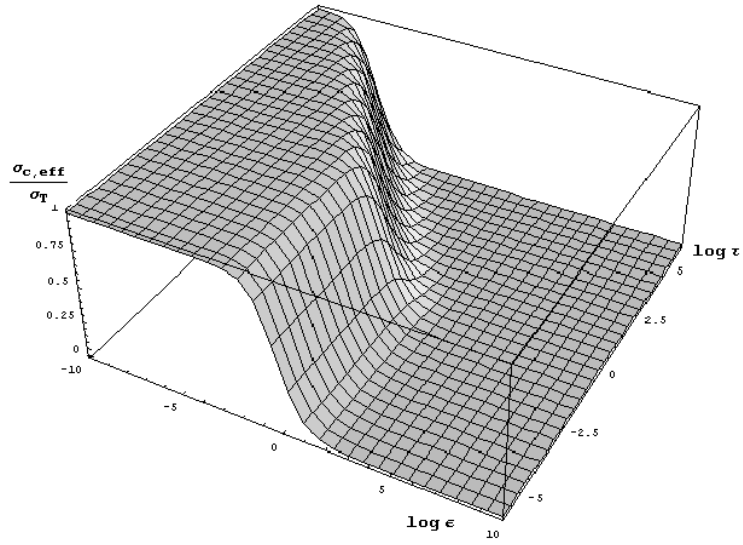


Figure 4: The effective Compton scattering cross section (normalized by Thomson cross section) as a function of $\log(\epsilon \equiv \frac{E}{m_e c^2})$ and $\log(\tau \equiv \frac{kT}{m_e c^2})$, where E and T are the photon energy and the temperature of non-degenerate gas of free electrons (see the text)

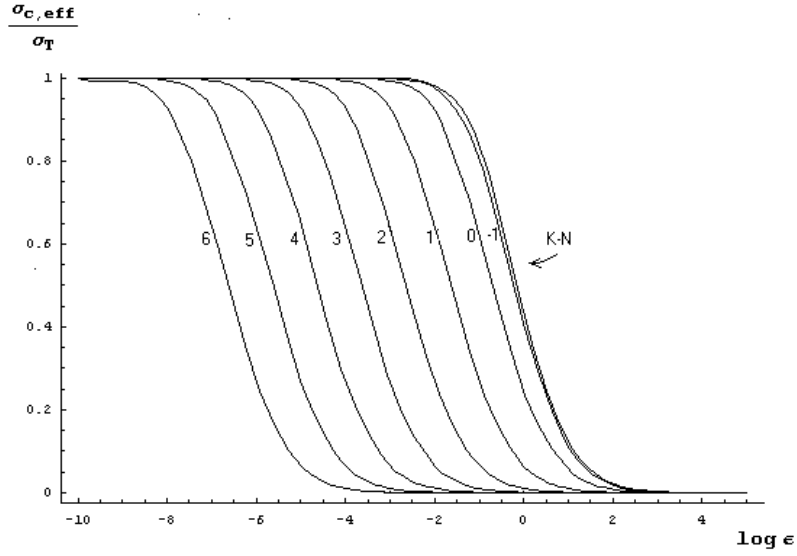


Figure 5: The effective Compton scattering cross section verses $\log(\epsilon \equiv \frac{E}{m_e c^2})$ (normalized by Thompson cross section). the label on each curve is the logarithm of $\frac{kT}{m_e c^2}$. In low temperatures $kT \ll m_e c^2$ the curve approach to the most right curve corresponding to the traditional Compton cross section as evaluated using Klain-Neshina formula (Fig.(1)). In higher an higher temperatures the overall behaviour is a shift toward left. (see the text)

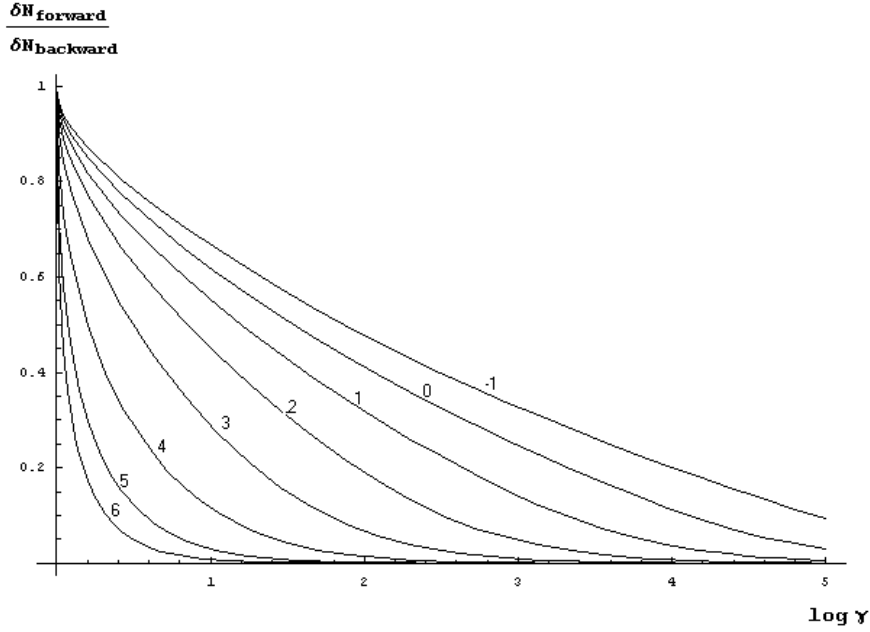


Figure 6: The ratio of photons scattered by the electrons moving in the same direction as the beam to ones moving in the opposite direction. The horizontal axis is the logarithm of electron Lorentz factor, and the number near each curve is the logarithm of $\epsilon \equiv \frac{E}{m_e c^2}$, with E being the incoming photon energy. It can be seen that the ratio equals one when electrons move slowly, while it approaches to zero for ultra-relativistic electrons, showing that in ultra-relativistic temperatures the electrons moving in the opposite direction have the most share in Compton scattering (Sec.4).

$\log \epsilon \backslash \log \tau$	-3	-2	-1	0	1	2	3
-1	0.997	0.976	0.813	0.399	0.110	0.019	0.003
0	0.991	0.924	0.626	0.244	0.056	0.009	0.001
1	0.930	0.648	0.265	0.065	0.011	0.002	0.0002
2	0.648	0.266	0.065	0.011	0.0015	0.00019	0.000024
3	0.265	0.064	0.0109	0.0015	0.000197	0.000024	2.83×10^{-6}
4	0.064	0.0109	0.0015	0.000197	0.000024	2.83×10^{-6}	3.26×10^{-7}
5	0.0109	0.0015	0.000197	0.000024	2.83×10^{-6}	3.26×10^{-7}	3.69×10^{-8}
6	0.0015	0.000197	0.000024	2.83×10^{-6}	3.26×10^{-7}	3.69×10^{-8}	4.13×10^{-9}

Figure 7: **TABLE I** The effective Compton cross section